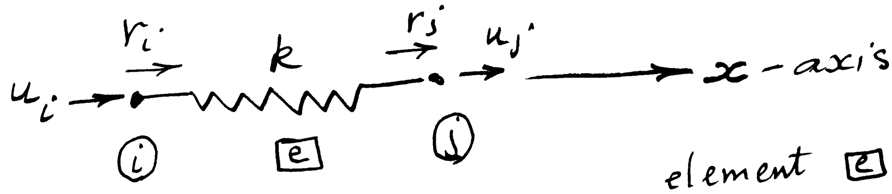
MENG 453

Dr. WIFI

Review # 1

- **Spring Element # linear 1-D**



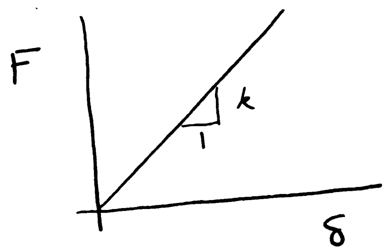
element $\boxed{e}$

nodes i, j

* 1 D.O.F. / Node

* 2 D.O.F / Element

* Axially Loaded # 1 D



$$F_{\boxed{e}} = k \cdot \delta_{\boxed{e}} \quad , \quad \delta_{\boxed{e}} = (u_j - u_i)_{\boxed{e}}$$

- Nodal displacement vector

$$\{d\}_{\boxed{e}} = \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_{\boxed{e}}$$

2X1

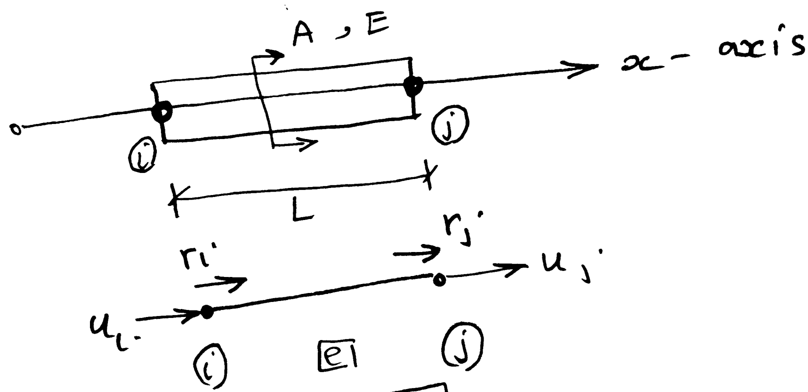
- Nodal Load vector

$$\{r\}_{\boxed{e}} = \begin{Bmatrix} r_i \\ r_j \end{Bmatrix}_{\boxed{e}}$$

- Element equilibrium

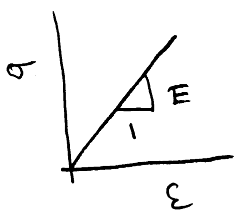
$$\{r\}_{\boxed{e}} = [K]_{\boxed{e}} \{d\}_{\boxed{e}} \quad , \quad [K]_{\boxed{e}} = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

• Bar Element # Linear, 1-D



$\boxed{k = EA/L}$

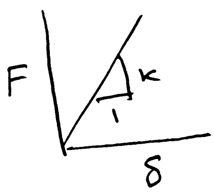
- 1 D.O.F / Node
- 2 D.O.F. / Element
- Axially loaded
- 1-D



$$\delta_{e} = (u_j - u_i)_{e}$$

$$\varepsilon_{e} = \frac{\delta}{L} = \left\lfloor -\frac{1}{L} \quad \frac{1}{L} \right\rfloor \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_{e}$$

$$\sigma_{e} = E \cdot \varepsilon_{e}$$



$$F_{e} = \sigma_{e} * A = \left(\frac{EA}{L}\right)_{e} \cdot \delta_{e} \rightarrow k = \frac{EA}{L}$$

- Element Equilibrium

$$\underset{2\times 1}{\begin{Bmatrix} r_i \\ r_j \end{Bmatrix}_{e}} = \left(\frac{EA}{L}\right)_{e} \underset{2\times 2}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}_{e}} \quad \text{or} \quad \{r\}_{e} = [k]_{e} \{d\}_{e}$$

$$\mathbb{r} = \mathbb{k}_{e} \cdot$$

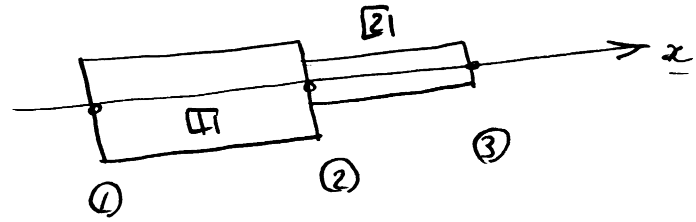
- internal nodal forces

$$\begin{Bmatrix} f_i \\ f_j \end{Bmatrix}_{e} = - \begin{Bmatrix} r_i \\ r_j \end{Bmatrix}_{e} \quad \text{or} \quad \underset{2\times 1}{\{f\}_{e}} = - \underset{2\times 1}{\{r\}_{e}}$$

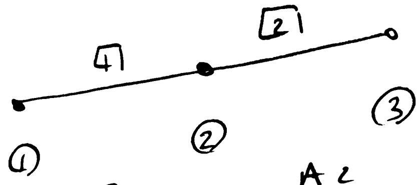
$$\mathbb{f} = - \mathbb{r}$$

• Assembly and Global (System) Equilibrium

Real system



Discretized system ≡ Model



A_1	A_2
E_1	E_2
L_1	L_2
$k_1 = \dfrac{E_1 A_1}{L_1}$	$k_2 = \dfrac{E_2 A_2}{L_2}$

Elem. #1

$$\begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix}_{\boxed{1}} = \frac{E_1 A_1}{L_1} \begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Elem. #2

$$\begin{Bmatrix} r_2 \\ r_3 \end{Bmatrix}_{\boxed{2}} = \frac{E_2 A_2}{L_2} \begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

• Global Load vector (nodal)

$$\underset{3\times1}{\mathbb{F} = \{F\}} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} r_{1\,\boxed{1}} \\ r_{2\,\boxed{1}} + r_{2\,\boxed{2}} \\ r_{3\,\boxed{2}} \end{Bmatrix} \quad \text{or} \quad F_i = \sum_{\substack{\text{at} \\ \text{node} \\ (i)}} r_{i\,\boxed{j}}$$

$j = 1 \sim$ no. of element connected at node ⓘ

• Global Stiffness Matrix

$$\underset{3\times3}{\mathbb{K} = [K]} = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \qquad [K] = \sum_{e=1}^{\text{no. of elements}} [k]_{\boxed{e}}$$

• Global Equilibrium

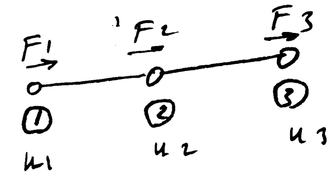
$$\mathbb{F} = \mathbb{K} \cdot \mathbb{U}, \qquad \mathbb{U} \equiv \{U\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

≡ Global nodal displacement vector

- Treatment of B.C. → Elimination

Let,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$



- Before imposing B.C. [K] is SINGULAR due to possible Rigid Body Motion (RBM)
- There MUST be enough B.C's to prevent RBM

EX. • Let $u_1 = \bar{u}_1$ (prescribed displacement, u_1 may be zero)

Knowns	Unknowns
$u_1 = \bar{u}_1$	$F_1 \to R_1$ reaction
F_2	u_2
F_3	u_3

- Elimination Gives

$$\begin{Bmatrix} F_2 - k_{21}\bar{u}_1 \\ F_3 - k_{31}\bar{u}_1 \end{Bmatrix} = \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

\# Solve for u_2 and u_3

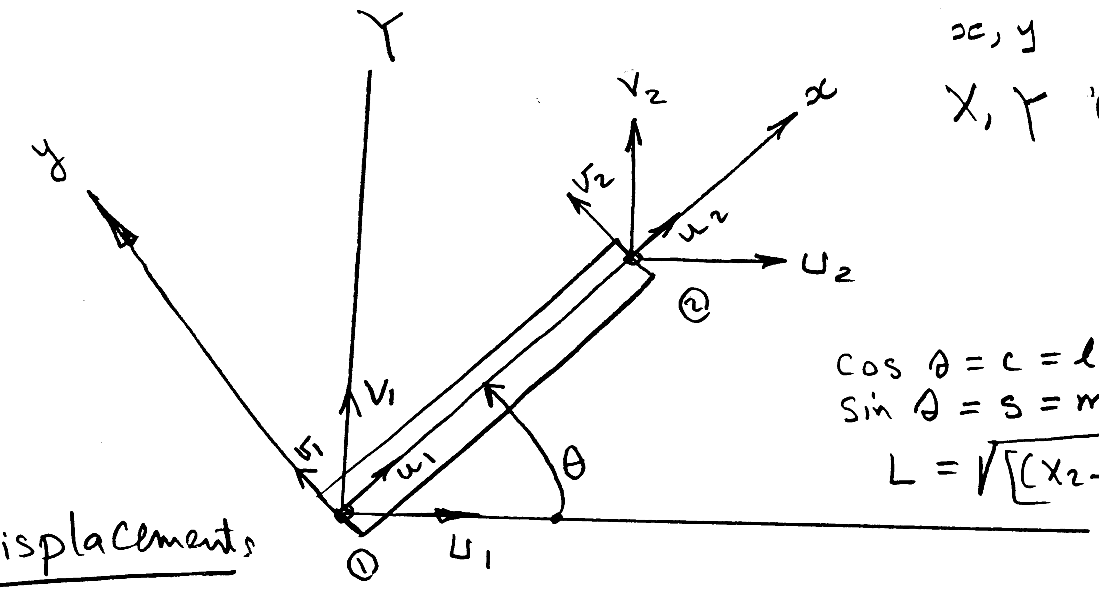
\# Reaction $R_1 = k_{11}\bar{u}_1 + k_{12}u_2 + k_{13}u_3$

- Global Equilibrium used to check analysis

$$R_1 + F_2 + F_3 = 0$$

TRUSS – 2D element

x, y Local coordinates
X, Y Global coordinates



$$\cos\theta = c = l = (X_2 - X_1)/L$$
$$\sin\theta = s = m = (Y_2 - Y_1)/L$$
$$L = \sqrt{[(X_2-X_1)^2 + (Y_2-Y_1)^2]}$$

Displacements

$$\{d\}_{local} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}, \quad \{d\}_{global} = \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix}$$

2 D.O.F. /node
4 D.O.F. /Element , 2 nodes / element
Axially Loaded – 2D

• Local element matrices

$$\underset{4\times1}{\begin{Bmatrix} r_1 \\ s_1 \\ r_2 \\ s_2 \end{Bmatrix}} = \underset{4\times4}{[K]_{local}} \underset{4\times1}{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}} \quad \text{(or)} \quad \boxed{\mathbb{F}_{local} = \mathbb{K}_{e\,local} \cdot \mathbb{d}_{local}}$$

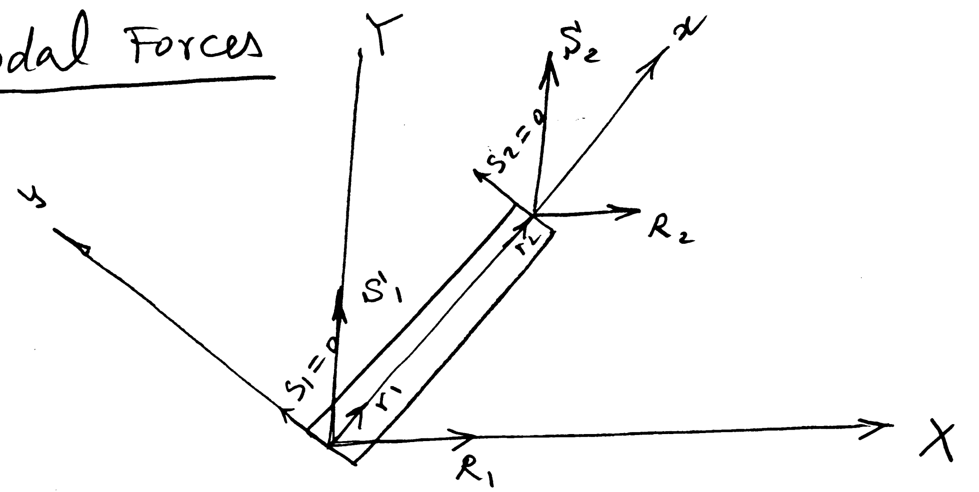
• NOTE ⇒ axially loaded member, $s_1 = s_2 = 0$, $v_1 = v_2 = ?$

$$\mathbb{K}_{e\,local} = [K]_{e\,local} = k_e \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}, \quad k_e = \left(\frac{EA}{L}\right)_e$$

• Transformations

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix} \quad \text{(or)} \quad \underset{4\times1}{\{d\}_{local}} = \underset{4\times4}{[T]} \underset{4\times}{\{d\}_{glob}}$$

• Nodal Forces



$$\{r\}_{local} = \begin{Bmatrix} r_1 \\ \cancel{s_1} \\ r_2 \\ \cancel{s_2} \end{Bmatrix}, \quad \{r\}_{global} = \begin{Bmatrix} R_1 \\ S_1 \\ R_2 \\ S_2 \end{Bmatrix}$$

• Transformations

$$\{r\}_{local} = [T]\{r\}_{global}$$

• Element Equibrium

$$\{r\}_{local} = [K]_{e_1\,local} \cdot \{d\}_{local}$$

$$[T]\{r\}_{global} = [K]_{e\,local} \cdot [T]\{d\}_{global}$$

$$\therefore \{r\}_{global} = \underbrace{\left[[T]^{-1}[K_{e}]_{local}[T]\right]}_{[K_{e_1}]_{global}} \cdot \{d\}_{gobal}$$

or simply

• $\{r\} = [K]_{e_1}\{d\}$, $\{r\} \equiv \{r\}_{global}$ & $\{d\} \equiv \{d\}_{gl}$

• $[K]_{e_1} = [T]^{-1}[K_e]_{local}[T]$, $[T]^{-1} = [T]^{*}$

Global Equilibrium

$$\{F\} = [K]\{U\}$$
$$n\times 1 \qquad n\times n \qquad n\times 1$$

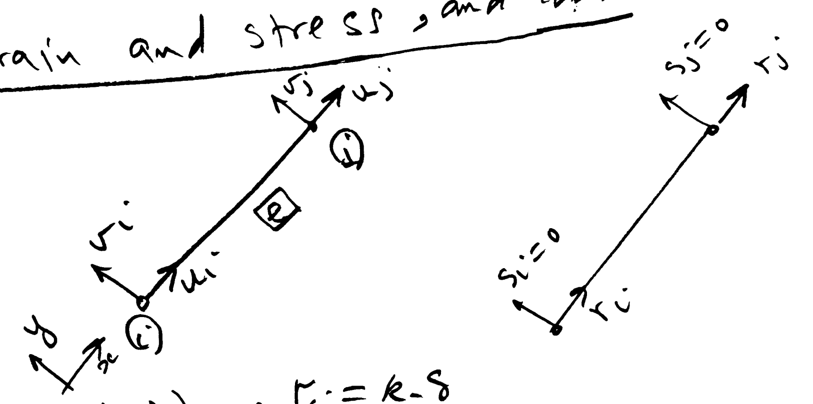
$$\{F\} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ \vdots \\ F_{xn} \\ F_{yn} \end{Bmatrix}, \quad \{U\} = \begin{Bmatrix} U_1 \\ V_1 \\ \vdots \\ U_n \\ V_n \end{Bmatrix}$$

$$F_{x_i} = \sum_j R_i, \quad j = 1 \sim \text{no. of elements connected at node } i$$

$$F_{Y_i} = \sum_j S_i, \quad j = \text{" \quad " \quad "}$$

Deformation, strain and stress, and load

$$k = \left(\frac{EA}{L}\right)_e$$



$$\delta = (u_j - u_i), \quad r_i = k\cdot\delta$$

$$\varepsilon \equiv \varepsilon_x = \delta/L = \left\lfloor -\tfrac{1}{L} \quad \tfrac{1}{L} \right\rfloor \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

$$= \frac{1}{L}\lfloor -1, 0, 1, 0 \rfloor \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}_{\text{lin}}$$

$$= \frac{1}{L}\underbrace{\lfloor -1, 0, 1, 0 \rfloor}_{1\times 4}\underbrace{[T]}_{4\times 4}\underbrace{\begin{Bmatrix} U_i \\ V_i \\ U_j \\ V_j \end{Bmatrix}}_{4\times 1}$$

$$\varepsilon_x = \frac{1}{L}\lfloor -c, -s, c, s \rfloor \{d\}_{\text{global}}$$

$$\sigma \equiv \sigma_x = \frac{E}{L}\lfloor -c, -s, c, s \rfloor \{d\}$$

$$F_{(e)} = \sigma_x A = \text{axial element load} = r_{i_{(e)}} = -r_{j_{(e)}}$$

- Remarks

Ⓘ $[T]^{-1} = [T]^{*}$

$$[T]^{-1}[T] = [I] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ⓘⓘ
$$[K]_{e} = [T]^{*}[k_{e}][T] = \left(\frac{EA}{L}\right)_{e} \begin{bmatrix} [\alpha] & [-\alpha] \\ [-\alpha] & [\alpha] \end{bmatrix}$$

$$[\alpha] = \overset{\text{local}}{\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}} \equiv \begin{bmatrix} l^2 & lm \\ ml & m^2 \end{bmatrix}$$

or

$$[K]_{e} = \left(\frac{EA}{L}\right)_{e} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

properties

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{i1} & K_{i2} & \cdots & K_{in} \\ \vdots & & & \\ K_{n1} & \cdots & \cdots & K_{nn} \end{bmatrix}$$

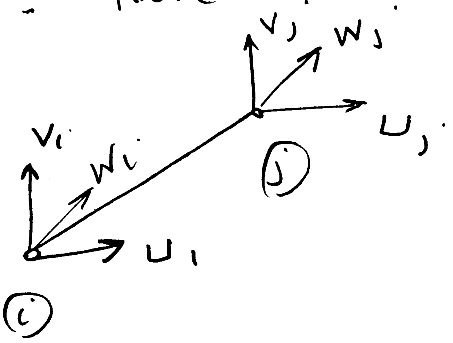
- symmetric, $K_{ij} = K_{ji}$
- $\sum_{i=1}^{n} K_{ij} = \sum_{j=1}^{n} K_{ij} = 0$
- $K_{ii} > 0$
- $[K]$ is singular before imposing B.C's
- There MUST be enough B.C's to prevent (RBM)

Ⓘⓘⓘ Assembly is only meaningful in GLOBAL coordinate system, leading to GLOBAL EQUI'M.

TRUSS - 3D Element

A three-dimensional truss (bar) element is processed in a similar way to the previous 2-D case. Here we have

- 3 D.O.F./node
- 6 D.O.F./element
- 2-nodes/element



- Define, the direction cosines

$$\ell = (X_j - X_i)/L$$
$$m = (Y_j - Y_i)/L$$
$$n = (Z_j - Z_i)/L$$

$$L = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2 + (Z_j - Z_i)^2}$$

- Transformation

$$\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{bmatrix} \ell & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell & m & n \end{bmatrix} \begin{Bmatrix} U_i \\ V_i \\ W_i \\ U_j \\ V_j \\ W_j \end{Bmatrix}$$

$[K]_{(e)}$ = global element stiffness matrix

$$= \left(\frac{EA}{L}\right)_{(e)} \left[\begin{array}{c|c} [\alpha] & [-\alpha] \\ \hline [-\alpha] & [\alpha] \end{array}\right]$$

$$[\alpha] = \begin{bmatrix} \ell^2 & \ell m & \ell n \\ m\ell & m^2 & mn \\ n\ell & nm & n^2 \end{bmatrix}$$